# Recurrence Relations

In this lecture, we show how to formulate a counting problem in terms of recurrence relations and then discuss methods of solving several common types of recurrence relations.

## 1. Introduction

**Definition**
Consider a sequence $\{a_n\}$. A **recurrence relation** for the sequence is an equation that relates the general term $a_n$ with some of its preceding terms $a_0, a_1, \ldots, a_{n-1}$.
For example, consider the sequence

$$\begin{cases} a_n = 1 + 2 + 2^2 + \cdots + 2^{n-1}; & r \geq 1 \\ a_0 = 0 \end{cases}$$

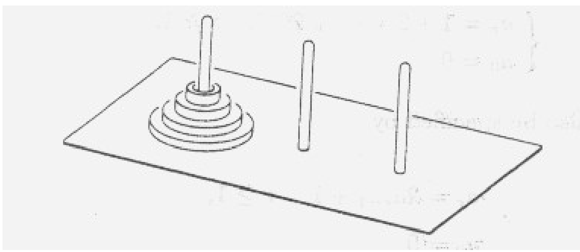
This sequence can also be specified by $a_n = 2a_{n-1} + 1$, $r \geq 1$, with $a_0 = 0$.
This is an example of a recurrence relation with **initial condition**. Depending on the nature of the problem, the sequence $a_n$ need not always start with the term $a_0$, that is $a_0$ may not have any physical meaning. In some texts, a recurrence relation is also called a **difference equation**. We shall use the abbreviation RR for recurrence relation.

Let us consider some example of formulating counting problems in terms of RR.

**Example 1.1**
Find a recurrence relation for $a_n$, the number of ways of arranging n distinct objects in a row.

Solution:
There are n ways of choosing an object to be placed in the first position of the row. After placing an object in the first position, the number of ways of arranging the remaining $(n - 1)$ objects is $a_{n-1}$.
Thus we have the RR $a_n = na_{n-1}$.
Clearly, we have $a_1 = 1$.
We can easily solve this RR to get $a_n = n!$, a well-known formula.

**Example 1.2 (The Tower of Hanoi)**
The Tower of Hanoi is a neat little puzzle invented by the French mathematician Édouard Lucas in 1883.
We are given a tower of r disks, initially stacked in decreasing size on one of three pegs. The objective is to transfer an entire tower to one of the other pegs, moving only one disk at a time and never moving a larger one onto a smaller. What is the minimum number of moves required?



Solution:
Let $T_n$ be the minimum number of moves that will transfer n disks from one peg to another under Lucas's rules.
It is advantageous to look at small cases first. It is easy to see that $T_1 = 1$ and $T_2 = 3$.
Here is one of the methods to transfer n $(n \geq 2)$ disks:
We first transfer the $n - 1$ smallest to a different peg (requiring at least $T_{n-1}$ moves), then move the largest (requiring one move), and finally transfer the $n - 1$ smallest back onto the largest (requiring at least another $T_{n-1}$ moves). It follows that $T_n \leq 2T_{n-1} + 1$.
We now show that there can be no shorter way. No matter what method we use to transfer the disks, at some point we must move the largest disk. When we do, then the other $n - 1$ smaller disks must be on a single peg, and it must have taken at least $T_{n-1}$ moves to put them there. After moving the largest disk, we must transfer the $n - 1$ smaller disks (which must again be on a single peg) back onto the largest; this too requires at least $T_{n-1}$ moves. Hence $T_n \geq 2T_{n-1} + 1$.
Thus, we have the RR $T_n = 2T_{n-1} + 1$ with initial condition $T_1 = 1$.
We shall show later that $T_n = 2^n - 1$ for $n = 1, 2, \ldots$.

**Example 1.3 (Lines in the plane)**
How many slices of pizza can a person obtain by making n straight cuts with a pizza knife?

Solution:
The given problem is equivalent to "What is the maximum number $L_n$ of regions defined by n lines in the plane?".
The Swiss mathematician Jacob Steiner first solved this problem in 1826.
By looking at small cases, we have $L_1 = 2$, $L_2 = 4$.
It can be observed that the n th line hits the previous (n – 1) lines in (n – 1) different places, hence it splits n old regions and so the number of regions increased by n.
Therefore, we have the RR $L_n = L_{n-1} + n$ for $n \geq 2$ with $L_1 = 2$.
Again we shall show later that $L_n = 1 + \binom{n+1}{2}$ for n = 1, 2, ....

**Example 1.4**
Let $a_n$ be the number of ways of climbing a staircase of n stairs, where each step can cover either one stair or two stairs. Find a RR for $a_n$.

Solution:
We see that $a_1 = 1$ and $a_2 = 2$.
For the general case of n stairs, consider the following two possibilities:
(1) If the first step covers one stair then there are $a_{n-1}$ ways to climb the remaining n – 1 stairs.
(2) If the first step covers two stair then there are $a_{n-2}$ ways to climb the remaining n – 2 stairs.
Thus we have $a_n = a_{n-1} + a_{n-2}$ for n = 3, 4, ...

## 2. Linear Recurrence Relation with Constant Coefficients

Consider a recurrence relation (RR) of the following form:
E: $a_n = Aa_{n-1} + Ba_{n-2} + f(n)$
where A, B are constants.
This is called a linear RR with constant coefficients with order 2.

## Method of Characteristic Equation

In the RR (E), if f(n) = 0 for all n then the RR is said to be **homogeneous**; otherwise it is said to be non-homogeneous. Let us write H: $a_n = Aa_{n-1} + Ba_{n-2}$

**Solution of Homogeneous Equation**
All possible solution of (H) can be determined using the characteristic equation $x^2 = Ax + B$:
(1) If $x^2 = Ax + B$ has distinct roots $\alpha$ and $\beta$ then $a_n = K\alpha^n + L\beta^n$, where K and L are constants.
(2) If $x^2 = Ax + B$ has the repeated root $\alpha$, then $a_n = (K + nL)\alpha^n$, where K and L are constants.

**Remark:**
The theorem concerns the general solution of a homogeneous linear RR with constant coefficient of order 2. The general solution of homogeneous RR of order $\neq 2$ can be determined similarly.

**Example 2.1**
Find the general solution of RR $a_n = 3a_{n-1} - 2a_{n-2}$ where $a_1 = 1$ and $a_2 = 2$.

Solution:
The characteristic equation is $x^2 - 3x + 2 = 0$ with roots 1 and 2.
$\therefore$ $a_n = K + 2^nL$
The initial conditions force $1 = K + 2L$
$2 = K + 4L$
That is, K = 0 and L = 1/2.
Hence $a_n = 2^{n-1}$.

**Example 2.2**
Consider the RR $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for n = 2, 3, ...

Solution:
The characteristic equation is $x^2 - x - 1 = 0$ with roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
We have $a_n = K\alpha^n + L\beta^n$
From the initial conditions $a_0 = 0$ and $a_1 = 1$,

$$0 = K + L$$
$$1 = K\alpha + L\beta$$

Solve for K and L, we have $K = 1/(\alpha - \beta) = 1/\sqrt{5}$ and $L = 1/(\beta - \alpha) = -1/\sqrt{5}$.

Hence $$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

**Solution of Non-homogeneous Equation**
E: $a_n = Aa_{n-1} + Ba_{n-2} + f(n)$ where A, B are constants.
The general solution of (E) can be found as follows:
Step 1: Find the general solution of the homogeneous part of (E), denoted by H.S.
i.e. (H): $a_n = Aa_{n-1} + Ba_{n-2}$
Step 2: Find a particular solution (P.S.) of (E).
Step 3: The sum H.S. + P.S. is then the general solution of (E).
To see this, let $a_n$ be any solution of (E). Then it is easy to show that $a_n$ – P.S. is a solution of (H).
Thus $a_n$ – P.S. = H.S. and so $a_n$ = P.S. + H.S.

**Method for finding a particular solution of a non-homogeneous RR**
There is no general procedure for determining a particular solution (P.S.) of a RR. However, if f(n) takes a simple form, we can use the method of inspection (or guessing). For example,
(1) f(n) = C, a constant.
Then we guess the particular solution P.S. = A, also a constant.
(2) If f(n) is of the form $P(n)\alpha^n$, where P(n) is a polynomial of degree k, then we guess a particular solution P.S. as follows:
(a) If $\alpha$ is not a root of the characteristic equation (of the homogeneous part), let
P.S. = $Q(n)\alpha^n$
where Q(n) is some polynomial of degree k.
(b) If $\alpha$ is root of the characteristic equation of multiplicity m, let
P.S. = $n^m Q(n)\alpha^n$
where Q(n) is some polynomial of degree k.

**Example 2.3**
Consider the RR $a_n + a_{n-1} = 3n2^n$.

Solution:
Let a P.S. be of the form $(cn + d)2^n$
Substituting it into the given RR, we obtain

$$(cn + d)2^n + (c(n-1) + d)2^{n-1} = 3n2^n$$
$$\Rightarrow \quad 3cn + (3d - c) = 6n$$
$$\Rightarrow \quad c = 2 \text{ and } d = 2/3$$

A P.S. of the RR is $(2n + 2/3)2^n$.

**Example 2.4**
Consider the RR $a_n = 4a_{n-1} - 3a_{n-2} + (n + 6)3^n$.

Solution:
The characteristic equation is $x^2 - 4x + 3 = 0$ with roots 3 and 1.
Hence the general solution of the homogeneous part is $K + L3^n$.
Since 3 is a root of the characteristic equation, a P.S. is $n(cn + d)3^n$
Substituting this P.S. into the RR, we get

$$n(cn + d)3^n = 4(n-1)[c(n-1) + d]3^{n-1} - 3[c(n-2) + d]3^{n-2} + (n + 6)3^n$$
$$\Rightarrow \quad c = 3/4 \quad \text{and} \quad d = 9$$

A P.S. is therefore $n(3n/4 + 9)3^n$.
Hence the general solution of the RR is $a_n = K + L3^n + n(3n/4 + 9)3^n$.

## 3. Method of Generating Function

**Example 3.1**
Consider the RR

$$a_n - 3a_{n-1} = 2 \quad \text{with boundary condition } a_0 = 1.$$

We multiply both sides by $z^n$ and sum over n = 1, 2, ... to get

$$\sum_{n=1}^{\infty} a_n z^n - 3\sum_{n=1}^{\infty} a_{n-1} z^n = \sum_{n=1}^{\infty} 2z^n$$

$$\Rightarrow \quad (A(z) - a_0) - 3z\,A(z) = \frac{2z}{1-z}$$

$$\Rightarrow \quad A(z) = \frac{1+z}{(1-3z)(1-z)} = \frac{2}{1-3z} - \frac{1}{1-z}$$

$$\Rightarrow \quad a_n = 2(3)^n - 1 \quad \text{for } n = 0, 1, 2, \ldots$$

In general, we multiply the given RR by $z^n$ and then sum over all r to get an equation involving the generating function A(z). By simplifying the equation, we would be able to express A(z) in terms of a rational function in z. We then find the coefficient of $z^n$ from the rational function.

**Example 3.2**
Consider the RR derived in the Tower of Hanoi Problem.
Here $T_0 = 0$, $T_n - 2T_{n-1} = 1$ for n = 1, 2, ...
Multiply by $z^n$ and sum over $n \ge 1$, we get $(A(z) - T_0) - 2zA(z) = \dfrac{z}{1-z}$

$$\therefore \quad A(z) = \frac{z}{(1-2z)(1-z)} = \frac{1}{1-2z} - \frac{1}{1-z} = \sum_{n\ge 0} 2^n z^n - \sum_{n\ge 0} z^n. \quad \text{Thus} \quad T_n = 2^n - 1.$$

**Example 3.3**
Consider the RR $L_0 = 1$, $L_n = L_{n-1} + n$ for n = 1, 2, ...
We get $(A(z) - L_0) = zA(z) + \sum_{n\ge 1} nz^n$,

$$\text{or} \quad A(z) = \frac{1}{1-z} + \frac{z}{(1-z)^3} = \sum_{n\ge 0} z^n + \sum_{n\ge 0}\binom{2+n}{2} z^{n+1}.$$

Hence $L_n = 1 + \binom{n+1}{2}$ for n = 0, 1, 2, ....

**Example 3.4**
Consider the RR $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for n = 2, 3, ...

$$\sum_{n\ge 2} a_n z^n = \sum_{n\ge 2} a_{n-1} z^n + \sum_{n\ge 2} a_{n-2} z^n$$

$$(A(z) - a_1 z - a_0) = z(A(z) - a_0) + z^2 A(z)$$

$$A(z) = \frac{z}{1-z-z^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha z} - \frac{1}{1-\beta z}\right) \quad \text{where} \quad \alpha = \frac{1+\sqrt{5}}{2}, \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

$$\text{Hence} \quad a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$